Problems for Masters proficiency exam in Algebra

April 16, 2026

The Masters proficiency exam in Algebra will consist of 8 problems from this list. The list is subject to change, the committee will make every effort to keep it stable, but there is no guarantee. Make sure that you are looking at the latest version.

1. Let G be a group and let $a, b \in G$.
   (a) Prove that there is an automorphism $\phi : G \to G$ such that $\phi(ab) = ba$.
   (b) Prove that ab and ba have the same order.

2. List without repetition all abelian groups of order 600 which have at least one element of order 100.

3. Let G be an abelian group having elements of orders 8 and 18.
   (a) Prove that G has an element of order 72.
   (b) Show by example that G need not have an element of order $8 \cdot 18 = 144$.

4. What is the largest order of an element of the alternating group A_9?

5. Suppose that H is a subgroup of G and let $D = \{(h, h) \mid h \in H\}$. Then D is a subgroup of the Cartesian product $G \times G$. Prove that D is a normal subgroup of $G \times G$ if and only if H is contained in the center of G.

6. (a) If $f : \mathbb{Z}_{15} \to \mathbb{Z}_5 \oplus \mathbb{Z}_{10}$ is a nontrivial group homomorphism, what is the kernel of f?
   (b) How many group homomorphisms $g : \mathbb{Z}_5 \oplus \mathbb{Z}_{10} \to \mathbb{Z}_{15}$ are there?

7. Let G be a finite group and p a prime divisor of $|G|$.
   (a) Suppose N is a normal p-subgroup of G. Prove that N is contained in every p-Sylow subgroup of G.
   (b) Prove that the intersection of all of the p-Sylow subgroups of G is the *largest normal* p-subgroup of G.

8. (a) Let $\phi : \mathbb{Z}_{39} \to S_7$ be a nontrivial group homomorphism. What subgroups of S_7 can be the image of ϕ? (There are many, but they can be described succinctly.)
   (b) Prove that there is no nontrivial group homomorphism $\psi : S_7 \to \mathbb{Z}_{39}$.

9. Let $\phi : G \to H$ be a homomorphism of groups and let N be a normal subgroup of G. Suppose that $\phi|_N : N \to H$ is an isomorphism. Show that G is the internal direct product of N and $\operatorname{Ker}\phi$. (That is, $G = N \oplus \operatorname{Ker}\phi$).

10. (a) Find the order $|G|$ of the group G given by all invertible matrices of the form $\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}$, $a, b \in \mathbb{F}_7$, where the group operation is matrix multiplication.
    (b) For each prime integer p that divides $|G|$, find the number of Sylow p-subgroups of G.

11. Let n be a positive integer, let G be a finite group with an odd number of elements, and let $\phi : S_n \to G$ be a homomorphism. Prove that ϕ must be trivial, i.e., $\phi(\sigma) = e$ for all $\sigma \in S_n$.

12. Give an example of a non-Abelian group of order 555.

13. Let G be a non-abelian group with $2n$ elements for an integer n.

    (a) If n is odd, show that all elements of G of order 2 are conjugate.
    (b) Give an example with n even where not all elements of G of order 2 are conjugate.

14. Prove that every group of order 56 has a normal subgroup of order 7 or order 8.

15. Let S_6 be the symmetric group on six letters. Determine whether the following elements of S_6 are squares (i.e., of the form σ^2):

    (a) $(1\,2\,3\,4)$.
    (b) $(1\,2\,3\,4\,5)$.
    (c) $(1\,2\,3)(4\,5)$.

16. Let G be a group and let $\operatorname{Aut} G$ denote the group of all automorphisms of G. If $g \in G$, the inner automorphism associated to g is the map $\phi_g : G \to G$ defined by $\phi_g(x) = gxg^{-1}$.

    (a) Prove that each $\phi_g \in \operatorname{Aut} G$.
    (b) Prove that the collection of all inner automorphisms is a normal subgroup of $\operatorname{Aut} G$.

17. Prove that no group of order 63 or 64 can be simple.

18. Let X be a nonempty finite subset of the group G. Prove that if X is closed under the group operation, then X is a subgroup.

19. A characteristic subgroup of G is a subgroup H such that $\phi(H) = H$ for every automorphism ϕ of G.

    (a) Suppose K is a characteristic subgroup of H and H is a normal subgroup of G. Prove that K is a normal subgroup of G.
    (b) Show by example that the conclusion of
    (c) is false if we only assume K is a normal subgroup of H.

20. Let S_n denote the symmetric group on n letters and let A_n denote the alternating subgroup.

    (a) Show that an element of S_n of odd order is always in A_n.
    (b) Conclude that S_n has more elements of even order than of odd order if $n \geq 4$.

21. Prove that if $|G| = 132$, then G is not simple.

22. Let H and K be subgroups of the group G and suppose K has finite index in G.

    Prove that $H \cap K$ has finite index in H.

23. Let G be a finite group in which every nontrivial element has order two. Prove that G is abelian and has the form $\mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$.

24. Let U be the smallest subgroup of the symmetric group S_4 containing all elements of the form $(a\ b)(c\ d)$ with $\{a, b, c, d\} = \{1, 2, 3, 4\}$. Is U a normal subgroup of S_4? Determine the order of U.

25. Is there a simple group of order 24?

26. List the finite groups that contain at most two conjugacy classes.

27. Prove that if H is a subgroup of G of finite index then there is a finite index normal subgroup of G, contained in H.

28. Prove that the intersection of two finite index subgroups of a group G has finite index in G

29. Prove that index is multiplicative: if H is a subgroups of a group G of finite index and K is a subgroup of H of finite index, then $[G : K] = [G : H] \cdot [H : K]$.

30. Find the order of $GL(2, \mathbb{F}_p)$.

31. How many 2-Sylow subgroups does S_4 have?

32. How many 3-Sylow subgroups does S_4 have?

33. Let A be a finite set, and let S_A be the symmetric group on A. Consider an action of S_A on the power set $\mathcal{P}(A)$ induced by the standard action of S_A on A (that is $\sigma \cdot B = \{\sigma(b) \mid b \in B\}$ for $\sigma \in S_A$ and $B \in \mathcal{P}(A)$.) How many orbits are there? For $B \in \mathcal{P}(A)$, describe its stabilizer.

34. Prove that if G is a group of order 12 then G either has a normal subgroup of order 4 or a normal subgroup of order 3.

35. Let G be affine maps of $\mathbb{C} \to \mathbb{C}$ of the from $z \mapsto az + b$, with $a \in \mathbb{C}^\times, b \in \mathbb{C}$.

    (a) Prove that G is a group with respect to the operation of composition of maps.

    (b) Let $H \subset G$ consist of the translations (that is $a = 1$ in the above). Prove that H is a normal subgroup of G, and that $G/H \cong \mathbb{C}^\times$.

36. Find the order of the multiplicative group $(\mathbb{Z}/100\mathbb{Z})^\times$.

37. Set $R = \mathbb{Z}[\sqrt{d}]$, where d is an integer that is not a perfect square. For $x = a + b\sqrt{d} \in R$, the map $x \mapsto C(x) = a - b\sqrt{d}$ is an automorphism of the integral domain R (*assume this*). Define $N(x) = x \cdot C(x) = a^2 - db^2$.

    (a) Prove that $N(xy) = N(x)N(y)$ for $x, y \in R$.

    (b) Show that x is a unit in R if and only if $N(x) = \pm 1$.

    (c) If $d < 0$, use part (b) to find all units in R.

38. Prove that every field $\mathbb{F}$ contains a subfield isomorphic to either $\mathbb{Q}$ or a finite field $\mathbb{F}_p$ of prime order.

39. Prove that the polynomial ring $\mathbb{Z}[x]$ is a unique factorization domain but not a principal ideal domain.

40. Prove or disprove: $\mathbb{Z}[\sqrt{2}]$ contains arbitrarily small nonzero elements.

41. Let R and S be commutative rings with 1.

    (a) Prove that if I and J are ideals of R and S respectively, then $I \times J$ is an ideal of the ring $R \times S$.

    (b) Prove that all ideals of the ring $R \times S$ have the form in (a).

42. Let $f, g \in \mathbb{Q}[x] \subset \mathbb{C}[x]$. Prove that the monic gcd of f, g in $\mathbb{Q}[x]$ equals the monic gcd of f, g in $\mathbb{C}[x]$.

43. Show that the subring of the complex numbers $\mathbb{Z}[\sqrt{-5}]$ is not a UFD (unique factorization domain).

44. Consider the map $\Phi : \mathbb{Q}[x] \to \mathbb{R}$ given by $\Phi(f) = f(\sqrt{2})$.

    (a) Show that Φ is a homomorphism and compute its kernel.

    (b) Show that the image of Φ is a subfield of $\mathbb{R}$ isomorphic to $\mathbb{Q}(\sqrt{2})$.

45. Factor 70 into irreducible factors in the Gaussian integers, $\mathbb{Z}[i]$.

46. Let X be the set of positive integers. Define a relation $\sim$ on X by $a \sim b$ if ab is a perfect square. Prove that $\sim$ is an equivalence relation.

47. Suppose that R is a commutative ring and I is a nonzero ideal of R that intersects the set of zero divisors of R only at 0. Prove that R is an integral domain.

48. (a) Show that $\mathbb{Z}_5[x]/(x^3 + x + 1)$ is a field.

    (b) Find the multiplicative inverse of the coset of $x + 1$ in $\mathbb{Z}_5[x]/(x^3 + x + 1)$.

49. Is $\mathbb{Z}[x]$ a Euclidean Domain?

50. Let $f : R \to S$ be ring homomorphism between commutative rings taking 1 to 1.

    (a) Prove that if P is a prime ideal of S, then $f^{-1}(P)$ is a prime ideal of R.

    (b) Show that the preimage of a maximal ideal does not have to be maximal.

51. Let R be a commutative ring with 1. Show that the *radical of R*

$$\sqrt{R} := \{r \in R \mid \exists n \in \mathbb{N} \text{ such that } r^n = 0\}$$

    is an ideal of R.

52. Consider the polynomial $x^2 + 1$ over the field $\mathbb{F}_7$. Prove that $\mathbb{F}_7[x]/(x^2 + 1)$ is a field of 49 elements.

53. Let R be a commutative ring with 1 which contains exactly one maximal ideal M. Prove that M is precisely the set of non-units of R.

54. Let $\phi : R \to S$ be a homomorphism of commutative rings and let M be a maximal ideal of R.

   (a) Prove that if ϕ is onto, then either $\phi(M) = S$ or $\phi(M)$ is a maximal ideal of S.

   (b) Give an example of (ϕ, R, S, M), ϕ not onto where $\phi(M)$ is not an ideal of S.

   (c) Give an example of (ϕ, R, S, M), ϕ not onto where $\phi(M)$ is an ideal of S which is not maximal.

55. Find all the units in the subring of the complex numbers $\mathbb{Z}[\sqrt{-5}]$.

56. Let R be a commutative ring. Recall that a non-zero R-module M is called *simple* or *irreducible* if it has only two submodules, 0 and M.

   (a) Show that if I is a maximal ideal of R, then R/I is a simple R-module.

   (b) Show that if M is a simple R-module, then there is a maximal ideal I of R with $M \cong R/I$.

57. Prove the following version of the Chinese Remainder Theorem for general rings (not necessarily commutative, possibly without 1). Let R be a ring with two-sided ideals I and J that are comaximal (i.e. $I + J = R$). Then

$$R/(I \cap J) \cong R/I \times R/J.$$

58. Let $p \in \mathbb{Z}$ be a prime integer. Prove that if $p \equiv 3 \bmod 4$, then p is irreducible in Gaussian integers $\mathbb{Z}[i]$.

59. Let R be a ring with 1. Recall that an element $r \in R$ is *nilpotent* if $r^n = 0$ for some positive integer n. Prove that if r is nilpotent then $1 - r$ is invertible.

60. Let R be a ring with 1. An element $e \in R$ is a *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$.

   (a) Prove that the subrings Re and $R(1 - e)$ have multiplicative identities.

   (b) Prove that $R \cong Re \times R(1 - e)$ as rings with 1.

61. Is the ideal generated by $f = (x^4 - 1)(x + 2)$ and $g = (x^2 - 4)(x + 1)$ in $\mathbb{Z}[x]$ principal?

62. Let R be a commutative ring. Recall that a non-zero R-module M is called *simple* or *irreducible* if it has only two submodules, 0 and M.

   (a) Prove that an R-homomorphism between two simple modules is either trivial or an isomorphism.

   (b) Show that for a simple module M the endomorphism ring $\mathrm{End}_R(M)$ is a division ring.

63. Which of the following polynomials are irreducible in $\mathbb{Q}[x]$?

   (a) $x^4 + 6x^3 + 9x^2 + 18x + 15$.

   (b) $x^3 - 3x^2 + 3x - 9$.

   (c) $x^2 + 3x + 9$.

64. Factor $f = x^4 - 2$ into a product of irreducible factors in the following rings.

(a) $\mathbb{C}[x]$.
(b) $\mathbb{Q}[x]$.
(c) $\mathbb{F}_7[x]$.

65. Factor the polynomial $f = x^3 + 6x^2 + 9x + 15$ completely over the following fields.

    (a) $\mathbb{Q}$.
    (b) $\mathbb{F}_2$.
    (c) $\mathbb{F}_5$.

66. Show that the following polynomials are irreducible over $\mathbb{Z}[x]$.

    (a) $f(x) = x^3 + 2x^2 - 3x + 1$.
    (b) $f(x) = x^4 - 4x^3 + 6$.

67. Factor the following polynomials into irreducible factors in the given rings.

    (a) $x^6 - 165$ in $\mathbb{Q}[x]$.
    (b) $x^4 - 1$ in $\mathbb{F}_{13}[x]$.
    (c) $x^9 + 1$ in $\mathbb{F}_3[x]$.

68. Prove that $\Psi_n(x) = x^{n-1} + \cdots + x + 1$ is an irreducible polynomial over $\mathbb{Q}$ if and only n is prime.

69. Factor the polynomial $x^4 + 5x + 1$ completely in the ring $\mathbb{F}_7[x]$.

70. Show that the following polynomials are irreducible in $\mathbb{Q}[x]$.

    (a) $x^3 + 2x^2 - 3x + 2$.
    (b) $x^4 - 4x^3 + 6$.

71. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$.

    (a) $x^3 - 3x + 1$.
    (b) $x^4 + 2x + 3$.
    (c) $x^5 - 12x^3 + 30x^2 + 18x + 15$.

72. Let $f = 2x^4 - 12x^3 + 30x + 6$. Factor f into irreducible factors in

    (a) $\mathbb{Q}[x]$.
    (b) $\mathbb{Z}[x]$.
    (c) $\mathbb{F}_5[x]$.

73. Which of the following polynomials are irreducible in $\mathbb{Q}[x]$?

    (a) $x^4 + 6x^3 + 9x^2 + 18x + 15$.
    (b) $x^3 - 3x^2 + 3x - 9$.
    (c) $x^2 + 3x + 9$.

74. For each of the following, determine whether the given polynomial is irreducible over the indicated field.

    (a) $x^3 - 2$ over the field with seven elements.

    (b) $x^4 + 2x^3 + x^2 - 1$ over the field of rational numbers.

    (c) $x^5 - 6x + 3$ over the field of rational numbers.

75. Find the gcd of $f = x^3 - 2$ and $g = x + 1$ in $\mathbb{Q}[x]$ and write it as a linear combination (in $\mathbb{Q}[x]$) of f and g.

76. Let p be a prime and n be a positive integer. Prove that $f(x) = x^n + p^{n+1}$ is irreducible in $\mathbb{Z}[x]$.

77. Does there exist a linear map $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathrm{Im}(\phi) = \mathrm{Ker}(\phi)$? If so, give an explicit example.

78. Let V be a finite dimensional real vector space equipped with a nonzero symmetric bilinear form $\langle\,,\rangle$. Show that the eigenvalues of the matrix representing the form in a basis of V depend on the choice of basis.

79. Let V be the vector space of all 2×2 real symmetric matrices and let $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

    (a) Show that the formula $T(A) = AX - XA$ defines a linear operator $T : V \to V$.

    (b) Find the matrix of T with respect to the following basis

$$\mathbf{v}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

80. Let $T : V \to V$ be a linear transformation and let x, y, z be eigenvectors for T with respective eigenvalues λ, μ, ν. Prove that if λ, μ, ν are distinct, then x, y, z are linearly independent.

81. Find all real symmetric matrices A such that $A^3 + A = 0$.

82. Suppose $v, w, v+w$ are all eigenvectors of the linear operator $\phi : V \to V$. Prove that $v, w, v+w$ all have the same eigenvalue.

83. Consider $\mathbb{R}^3$ with the usual inner product. Let $W = \{\, (x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0 \,\}$.

    (a) Find an orthonormal basis of W.

    (b) Extend the basis you found in (a) to an orthonormal basis of $\mathbb{R}^3$.

84. Let $\mathbb{F}$ be a field. Let $m \neq n$ be distinct positive integers and let A be an $m \times n$ matrix over $\mathbb{F}$ and B be an $n \times m$ matrix over $\mathbb{F}$ (so A, B are not square). Prove that it is impossible for both AB and BA to be identity matrices.

85. Let A be a real symmetric matrix. Prove that $A^2 + I$ is invertible.

86. Let A be a 4×4 complex matrix with minimal polynomial $x^3 - x^2$.

    (a) What are the possible characteristic polynomials of A?

    (b) What are the possible Jordan forms of A?

87. Suppose V, W are finite dimensional vector spaces and $\phi : V \to W$ is a surjective linear transformation. Prove that there is a linear transformation $\psi : W \to V$ such that $\phi \circ \psi = \mathrm{id}_W$.

88. Let V be a finite dimensional vector space and let A, B be subspaces of V. Prove that $\dim A + \dim B = \dim A \cap B + \dim(A + B)$.

89. Suppose V is a vector space over $\mathbb{F}$ and $T : V \to V$ is a linear transformation. Suppose that $v \in V$ and m is a positive integer such that $T^m(v) = 0$ but $T^{m-1}(v) \neq 0$. Prove that $v, T(v), \ldots, T^{m-1}(v)$ are linearly independent.

90. Suppose V, W are vector spaces over F and $T : V \to W$, $S : W \to V$ are linear transformations. Suppose also that $S \circ T$ is an isomorphism. Prove that W is the direct sum $\operatorname{Im} T \oplus \operatorname{Ker} S$.

91. Let A be the matrix $\begin{pmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ for $c \in \mathbb{C}$. What is the Jordan form of this matrix? (The answer may depend on c.)

92. Let V be a complex vector space and let $S, T : V \to V$ be two commuting operators $ST = TS$. Recall that a subspace $U \subset V$ is T-*invariant* if $T(U) \subset U$.

    (a) Prove that any eigenspace of S is T-invariant.

    (b) Conclude that S and T have a common eigenvector.

93. Let V be a vector space over a field. A linear operator T on V is said to be idempotent if $T^2 = T$. Prove that if T is idempotent then $V = \operatorname{Ker}(T) \oplus T(V)$.

94. Recall that a complex $n \times n$ matrix is said to be unitary if its complex conjugate transpose is equal to its inverse. Prove that an eigenvalue of a unitary matrix is of the form $e^{i\theta}$ for some real number θ.

95. Give an example of two matrices, A and B, that have the same characteristic polynomials and the same minimal polynomials, yet A and B are not similar. Assume we are working over the complex numbers or else please indicate the field of scalars that you are using.

96. Let $A = \begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}$. Find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

97. Prove that if A is an invertible complex matrix, then A^*A is Hermitian and positive definite.

98. Let A be a square matrix over $\mathbb{C}$. Prove that A is similar over $\mathbb{C}$ to its transpose A^T.

99. Prove that $SL_n(\mathbb{Z})$ has index 2 in $GL_n(\mathbb{Z})$.

100. Consider the standard action of $SO(3)$ on the unit sphere S^2 in $\mathbb{R}^3$.

    (a) Prove that the action is transitive.

    (b) Prove that a nontrivial element of $SO(3)$ has exactly 2 fixed points.