Masters Proficiency Exam in Statistics – August 2017

Note: There are six problems in this exam. Partial credit will be given for partial solutions but scoring will emphasize completely correct answers. You are guaranteed a passing score if you solve at least four problems. Show all work and justify all claims.

1. Let X be a random variable with pmf $f(k) = \binom{n}{k} p^k(1-p)^{n-k}$, $k \in \{1, \ldots, n\}$, with $n \in \mathbb{N}$ and $p \in (0,1)$.

   (a) Find the characteristic function (or moment generating function) of X.

   (b) Use (a) to find $E(X)$.

2. Let (X, Y) be a random vector with density

   $$f(x,y) = \frac{1}{2x^2}, \ (x,y) \in R, \text{ with } R = \{(x,y) : |x| < 1, \ 0 < y < x^2\}.$$

   Prove that $U = Y/X^2$ has a Uniform$(0,1)$ distribution.

3. Prove that a sequence of random variables that converges in probability is bounded in probability. (*Note:* $\{X_n\}_{n \geq 1}$ is bounded in probability if for every $\delta > 0$ there exists a $K_\delta > 0$ and a $N_\delta \in \mathbb{N}$ such that $P(|X_n| > K_\delta) < \delta$ for every $n \geq N_\delta$.)

4. Let $X_1, \ldots, X_n$ be i.i.d. with pdf

   $$f(x;\theta) = \frac{\theta}{(1+\theta x)^2}, \ x \geq 0, \text{ with } \theta > 0.$$

   (a) Find the estimating equation of the maximum likelihood estimator $\hat{\theta}_n$ (i.e. the equation whose solution is $\hat{\theta}_n$).

   (b) Find the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ (in particular, find the explicit form of the asymptotic variance).

   (c) Use (b) to derive an asymptotic confidence interval of level 95% for θ.

5. Let $X_1, \ldots, X_n$ be i.i.d. with Uniform$(0, \theta)$ distribution. Find a complete sufficient statistic for θ, and *prove* it is sufficient and complete.

6. Let $X_1, \ldots, X_n$ be i.i.d. with pdf $f(x;\theta) = \theta x^{\theta - 1}$, $0 < x < 1$, with $\theta > 0$.

   (a) Show that $\{f(x;\theta) : \theta > 0\}$ is a family with monotone likelihood ratio.

   (b) Find the uniformly most powerful test of level α for the hypotheses $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$, where θ_0 is a fixed given number.