MASTERS PROFICIENCY EXAM IN ALGEBRA
August 22, 2019

There are eight problems. Show all work. Be sure to justify all claims. Partial credit will be given for partial solutions, but scoring will emphasize completely correct answers. You are guaranteed a passing score if you successfully solve at least five problems.

(1) Let H and K be subgroups of the group G and suppose K has finite index in G. Prove that $H \cap K$ has finite index in H.

(2) List without repetition all abelian groups of order 600 which have at least one element of order 100.

(3) a) Find the order $|G|$ of the group G given by all invertible matrices of the form

$$\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}, \qquad a, b \in \mathbb{Z}_7,$$

where the group operation is matrix multiplication.
b) For each prime integer p that divides $|G|$, find the number of Sylow p-subgroups of G.

(4) Factor $f = x^4 - 2$ into a product of irreducible factors in the following rings.
(a) $\mathbb{C}[x]$.
(b) $\mathbb{Q}[x]$.
(c) $\mathbb{Z}_7[x]$.

(5) Find all the units in the subring of the complex numbers

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a,\, b \in \mathbb{Z}\}.$$

Make sure to prove your answer.

(6) Let I, J be ideals of the ring R with $I + J = R$ and $I \cap J = \{0\}$. Prove that $R \cong R/I \times R/J$.

(7) Let V be a finite dimensional vector space and $T : V \to V$ a linear operator. Prove that T is one-to-one if and only if T is onto.

(8) Let $T : V \to V$ be a linear transformation and let $x,\, y,\, z$ be eigenvectors for T with respective eigenvalues $\lambda,\, \mu,\, \nu$. Prove that if $\lambda,\, \mu,\, \nu$ are distinct, then $x,\, y,\, z$ are linearly independent.