MASTERS PROFICIENCY EXAM IN ANALYSIS

August 2019

There are 6 questions on this exam. You should attempt to complete all the questions. Show all your work, and justify your answers fully. Unsupported answers will receive little or no credit. You have 3 hours to finish. Good luck.

1. Show that if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} x_n^2 = x^2$.

2. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). By considering the function
$$\phi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$
show that there is a point $\xi \in (a,b)$ at which
$$f'(\xi) = \frac{f(b)-f(a)}{b-a}.$$

3. Suppose that a function $f : [0,1] \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x \in (0,1)$. Moreover, suppose that $[f(x)]^2 = 2\int_0^x f(t)dt$ for all $x \in [0,1]$. Prove that $f(x) = x$ for all $x \in [0,1]$.

4. Prove the following. If $\{a_n\}$ and $\{b_n\}$ are two sequences of strictly positive numbers such that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq 1$, then the following hold:

   (a) if $\sum_n b_n$ converges, then $\sum_n a_n$ converges;

   (b) if $\sum_n a_n$ diverges, then $\sum_n b_n$ diverges.

5. Consider
$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
Show that

   (a) for each fixed $y \in \mathbb{R}$, the function g defined by $g(x) = f(x,y)$ is continuous on $\mathbb{R}$;

   (b) for each fixed $x \in \mathbb{R}$, the function h defined by $h(y) = f(x,y)$ is continuous on $\mathbb{R}$;

   (c) f is NOT continuous on $\mathbb{R}^2$.

6. Let $I := [0, \frac{\pi}{2}]$ and $f : I \to \mathbb{R}$ be defined by $f(x) := \max\{x^2, \cos x\}$ for $x \in I$. Show there exists an absolute minimum point $x_0 \in I$ for f on I. Show that x_0 is a solution to the equation $\cos x = x^2$.