Overview of R.H. Stockbridge's Research

The vast majority of my research has been connected to the relation between stochastic control and linear programming. Recent investigations have derived a tractable, nonlinear optimization problem from the linear program. This work can be characterized as being foundational and applied, with some efforts at approximation of these infinite-dimensional linear programs.

Foundational Research - Linear Programming

The foundational work began with my PhD dissertation, published in [2] and [3] in *The Annals of Probability.* We proved that long-term average stochastic control problems in which the control acts absolutely continuously in time are equivalent to linear programs over the space of stationary distributions for the controlled process. The process is characterized as a solution of a controlled martingale problem for its generator A. [2] establishes the existence of a stationary solution corresponding to each measure μ that satisfies the stationarity condition $\int Af\,d\mu = 0$ for all $f \in \mathcal{D}(A)$, while [3] utilizes this existence result to establish the equivalence. The proof merely shows existence of an adapted relaxed control without specifying its form.

Paper [11] significantly improved this equivalence in several ways. Most important is that to each stationary measure μ, a relaxed control of feedback type is identified as being given by the regular conditional distribution η on the control space U given the state x of μ; that is, the relation between μ and η satisfies $\mu(dx \times du) = \eta(x, du)\mu_E(dx)$ with μ_E being the state marginal. In addition, the equivalence between stochastic control problems and linear programming problems was extended (with this feedback control) so that it applied to discounted, first-exit, and finite-horizon problems as well as to long-term average problems. The key to this extension is to reformulate each of the problems as a long-term average problem for a process that reinitializes at the time of exit, or at the final time or, treating the discount factor as an exponential lifetime, at the time of death. Applying the existence result for long-term average problems and using a change in measure then leads to the desired equivalence.

Papers [19], [20] and [29] further extend these foundational results. In [19], the processes are characterized by a singular, controlled martingale problem which facilitates the modeling of singular behavior due to the underlying structure or due to the control action. An important aspect of this is a correct relaxation of the singular aspect of the model as a measure-valued random variable. The characterization of stationary measures, $\int Af\,d\mu_0 + \int Bf\,d\mu_1 = 0$ for all $f \in \mathcal{D}$, now includes two measures μ_0 and μ_1 with μ_0 being the stationary measure μ of [11] while μ_1 captures the stationarity of the singular aspects of the process. This paper proves the existence of a stationary solution of the martingale problem (with appropriate feedback controls η_0 and η_1) corresponding to any pair (μ_0, μ_1) satisfying the stationarity condition. [20] extends the equivalence between stochastic and linear programming formulations to optimal stopping problems and [29] extends the linear programming equivalence to include processes having singular behavior but not singular control actions.

The final steps for equivalence between singularly controlled processes and linear programs involve using the long-term average existence result of [19] much as in [11]. The technical challenges imposed by the singular control action are non-trivial. The paper [arXiv:1707.09209] proves the equivalence for long-term average and discounted criteria. A second paper which addresses first exit, finite horizon, and optimal stopping with control, both without and with discounting, is nearly complete. These two papers will be submitted for publication as companion papers.

The paper [40] utilized the equivalence between the stochastic control and linear programming problems to give very general sufficient closedness and compactness conditions on the martingale model that ensure the existence of an optimal control in the class of strict controls. This single set of conditions applies to many standard stochastic control models and seem to be the most general to-date. This condition is also adapted to singular control problems in [arXiv:1707.09209] and its companion to prove the existence of optimal strict absolutely continuous and singular control policies.

Foundational Research - Nonlinear Optimization

The papers [33], [37], [39], [47], [48] and [arXiv:1702.01041] (along with many conference papers since 2010) develop a new solution methodology based on the linear programming formulation for stochastic control problems. With regard to long-term average singular control problems, using a particular function f such that $Af = -1$ in the stationarity condition implies that $\int Bf\,d\mu_1 = 1$ for all feasible measures. Rewriting the cost criterion solely in terms of μ_1 then enables the infinite-dimensional problem to be reduced to a tractable, finite-dimensional nonlinear optimization problem. We initially developed this solution method for the optimal stopping of one-dimensional diffusions [33] in which the value function is constructed from the solutions of the nonlinear problem. This nonlinear optimization approach applies to all of the main decision criteria by carefully choosing f for the characterizing stationarity conditions. The papers [37], [39] and [47] apply this methodology to a variety of discounted stochastic control problems while the papers [48] and [arXiv:1702.01041] consider long-term average inventory problems without utilizing a linear programming formulation.

Approximation Research

Beginning with my PhD student's dissertation [12], a strong current in my research has been to focus on how to use the linear programming formulation to solve stochastic control problems. This effort necessarily entails approximating the infinite-dimensional linear programs by finite-dimensional linear programs. [12] gave conditions under which the values and the optimal controls for the approximating problems converge to the corresponding value and an optimal control for the infinite-dimensional problem.

Papers [16], [17], [18] and [24] characterize processes on a bounded region by their moments and rewrite the stationary condition on the measure μ as an infinite collection of conditions on their moments. In our first foray using this approach, [16] examines a controlled process and shows how the tightness (or lack thereof) of the constraints arising from a particular control give evidence of optimality (or not) of this control. We use the linear programming equivalence to estimate the moments related to the exit time distribution, the resolvent and the Laplace transform of uncontrolled processes in [17] and [18]. The novelty and importance of this method was recognized by [17] being published in *Operations Research.* In [24], an extension of the moment characterization to the simplex enabled this approach to analyze the moments related to the stationary distribution for the Wright-Fisher diffusion in population genetics. The paper [25] establishes the moment characterization for a general polytope and other bounded regions.

Typically, a finite difference scheme has been used to approximate solutions to stochastic control problems. The papers [30] and [32] examine a finite element approach to the solution of the linear program formulation of the problem and demonstrate much better accuracy than obtained from finite-difference schemes or dynamic programming. A current PhD student is expanding on the work in [32] with excellent results.

Applied Research

The linear programming formulation for stochastic control problems has been applied in a variety of settings. My early work [6] and [9] examined a new model for the wear of a machine in which the level of wear was given by the running maximum of a controlled diffusion process. These were among the earliest, if not the first, use of the running maximum in stochastic control theory. The paper [22] showed how to adapt the linear programming formulation when the cost was determined by the local time of a diffusion, with an application to the optimal control for an active shock absorber.

Employing the nonlinear optimization approach, [37] considered the optimal management of forests while [42] solved a different harvesting problem in which the optimal harvesting strategy requires the relaxed formulation of [19]. The paper [39] examined optimal production decisions.

Our most recent publications examine inventory management under both discounted [47] and long-term average ([48] and [arXiv:1702.01041]) criteria. Each paper extends the optimality of (s, S) ordering policies to general one-dimensional diffusion models. In particular, the submitted paper [arXiv:1702.01041] significantly relaxes the assumptions on the model and the cost structure resulting in the most general set of conditions under which optimality of an (s, S) policy can be proven for one-dimensional diffusion models. The nonlinear optimization method gives the solution without applying either the method of vanishing discount or the quasi-variational inequality. Surprisingly, weak convergence arguments are used without requiring the approximating singular measures to converge.

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