

# Días de Combinatoria: Peaks of Permutations

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As a student, the first math conference I ever attended was when I was one year from completing my Ph.D. Nowadays, I find that more students attend conferences and often presenting their own research, which is always super impressive. Yet, there is a type of conference that students might not know as much about. They are called “summer schools” and, as the name indicates, they consist of short introductory courses in topics often not encountered in the usual undergraduate curriculum. The sessions usually bring some known experts in a field, and include collaborative problem solving sessions, which are often run by graduate students who study in the area of the course. These summer schools often last a week or two and happen in a variety of locations across the world. The courses are often free and depending on funding, admitted students are provided with local shared accommodations and some meal support. I must remark that these programs are competitive and often occur in international locations. One place to learn more about summer school opportunities is to check out the AMS Calendar [4], where a lot of conferences and similar events are advertised.

I never attended a summer school as a student, but I had the opportunity to serve as a short course leader for Días de Combinatoria 2025 in Bogotá, Colombia. This event was a “Escuela de Matemática de América Latina y del Caribe (EMALCA)”, which translates to “School of Mathematics of Latin America and the Caribbean.” These schools are an initiative of the Mathematical Union of Latin America and the Caribbean (UMALCA), and their main objective is to contribute to the development of mathematics throughout Latin America and the Caribbean by connecting the “continent’s most prestigious mathematicians and researchers with undergraduate and graduate students. These schools also receive support from the International Center for Pure and Applied Mathematics (CIMPA)” [2].

It truly was an honor to get to present lectures on a topic of mathematics quite dear to me, peaks of permutations. Now, the main challenge for me was having to deliver these lectures in Spanish, because even though my mother-tongue is Spanish, I did all of my mathematical training in the United States in English. I struggled with the language at times, not knowing how to say “corollary” (which is corolario) or “counterexample” (which is contraejemplo), but the students were incredibly kind and did not get frustrated with me when I still mispronounced the words.

For students whose language is distinct than the host language, let me assure you that many if not all of the students that attend speak a decent amount of English. In fact, in summer 2024 multiple of my graduate students, some of whom do not speak Spanish, attended Encuentro Colombiano de Combinatoria (ECCO) and were able to communicate with participants through the two week event. It really is a great opportunity to build international collaborations, and to immerse themselves in a new culture all while learning new mathematics.

Now that you know about summer schools, such as Días de Combinatoria and Encuentro Colombiano de Combinatoria, let me share with you some of the math included in my short course. The math is followed with details on the structure and organization of the program.

## Permutations and their Peaks

Imagine you have the numbers 1, 2, 3 and you want to arrange them in a line. In how many ways can you arrange the numbers? Well, we could have

123, 132, 213, 231, 312, and 321.



Figure 1: Poster for Días de Combinatoria 2025 and photo at the start of the short course.

So, there are six ways to arrange the numbers. Notice that the first number had 3 options, the second 2 options, and the last number is exactly the number we had yet to use, so there is only a single option for that number. So, the total number of ways to arrange the numbers 1, 2, and 3 can be counted with the product  $3 \cdot 2 \cdot 1 = 6$ , as we expected. This idea holds more generally: If we want to count all the ways to arrange the numbers  $1, 2, 3, \dots, n$ , we first decide which number will go first and there are  $n$  options, then there are  $n - 1$  options for the second number and  $n - 2$  numbers for the third, and so on, until there is a single option for the last number. So, the total number of ways to arrange the numbers  $1, 2, \dots, n$  is given by the product  $n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1 = n!$ .

The set of all the ways of arranging the numbers  $1, 2, 3, \dots, n$  is called the set of permutations, and we denote this set with  $S_n$ , and each element in the set is called a permutation. As we saw above

$$S_3 = \{123, 132, 213, 231, 312, 321\}.$$

When  $n$  is small listing the numbers in order is fine, as we can tell the position of each number. But now imagine that we are trying to arrange the numbers  $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ . If I give you the permutation  $568741112109$ , it is unclear what is going on in the  $111$  part. Does this mean that the number 11 appeared as the 6th number and 1 as the 7th, or does it mean that 1 is the 6th number and 11 as the 7th? Now what if we had the numbers  $1, 2, 3, \dots, n$  and  $n$  is huge. You can imagine what a nightmare it would be to determine what someone means when placing the numbers side by side!

To combat this issue we introduce the notation  $\pi = \pi_1\pi_2\pi_3 \cdots \pi_n$  where  $\pi_i$  is the number that appears in position  $i$  in the permutation  $\pi$ . Using this notation we could clarify what we mean by  $\pi = 568741112109$ , by specifying that

$$\pi_1 = 5, \pi_2 = 6, \pi_3 = 7, \pi_4 = 8, \pi_5 = 7, \pi_6 = 4, \pi_7 = 11, \pi_8 = 1, \pi_9 = 2, \pi_{10} = 10, \text{ and } \pi_{11} = 9.$$

Another advantage of this notation is that we can use the information to plot the permutation. That is, on an  $xy$ -plane we plot the points  $(i, \pi_i)$  and we connect consecutive points with line segments. For example, when  $n = 3$  the plots for the six permutations are illustrated in Figure 2.

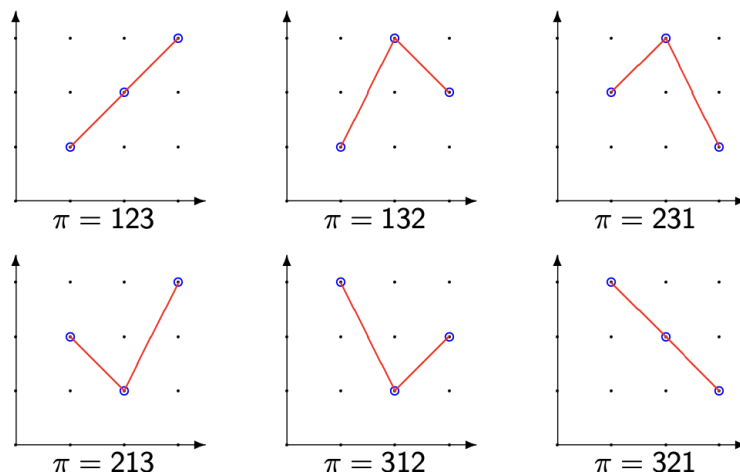


Figure 2: Plots of the permutations in  $S_3$ .

Looking at the plots one thing is curious. Some permutations have peaks, places they go up and down, while others do not. Here we need to be very careful to define what we mean by peaks. Using the notation  $\pi = \pi_1\pi_2 \cdots \pi_n$ , we say that the permutation  $\pi$  has a *peak at position  $i$*  if  $\pi_{i-1} < \pi_i > \pi_{i+1}$ . For example, 132 has a peak at position 2 because  $1 < 3 > 2$ , while 123 does not have a peak because even though  $2 < 3$  there is no number to the right of 3 with which to compare 3 to.

Get your hands dirty! Plot or write down all of the permutations  $S_4$  and ponder the following questions:

1. At what positions do the peaks appear?
2. How many permutations have no peaks?
3. How many permutations in  $S_4$  have peaks at position 2, how about at position 3?
4. Can a permutation have peaks at positions 2 and 3 simultaneously?

There has been a lot of mathematical research in understanding and counting the permutations that have peaks at specified locations. For us let's start small and consider what happens if there are no peaks at all!

For  $S_1 = \{1\}$ , the permutation 1 has no peaks. So, for  $n = 1$ , there is 1 permutation with no peaks. For  $S_2 = \{12, 21\}$ , both permutations have no peaks because there is not yet enough room to be able to have neighboring values both to the left and right. So, for  $n = 2$  there are 2 permutations with no peaks. For  $S_3$ , we can confirm with Figure 2 that the permutations 132, 213, 312, and 321 have no peaks. So, for  $n = 3$ , here are 4 permutations with no peaks. Having completed problem 2 above, you likely determined that there are 8 permutations in  $S_4$  that have no peaks.

We have determined that the number of permutations of  $S_n$  which have no peaks begins with the numbers 1, 2, 4, and 8. One might conjecture then that:

There are  $2^{n-1}$  permutations in  $S_n$  that have no peaks.

How would you begin to prove this? One idea is to think about where the number  $n$  must be if there are to be no peaks in the permutation. Then by spending some time staring at the permutations we found without peaks, you might deduce a way to start with a permutation in  $S_n$  that has no peaks and build a permutation in  $S_{n+1}$  that has no peaks. This will mean that you have to put the number  $n + 1$  somewhere making sure to not create a peak!



Figure 3: Días de Combinatoria, Bogotá Colombia, June 2025

Once you figure out that pattern and can confidently say that there are  $2^{n-1}$  permutations with no peaks, you might then consider how many permutations in  $S_n$  have a single peak at position  $i$ . First, begin with the peak being at position 2, and then at position 3, and then 4, and a pattern might emerge.

Of course there are many other possible collections of positions at which peaks could occur. In fact, the number of possible collections of where peaks occur is a very famous sequence of numbers, and you should see what it is! Now, if these concepts *peak* your interest (pun intended!), then you might want to read more about these topics. To start you might be interested in learning about the formula for counting the number of permutations in  $S_n$  with a fixed set of position at which they have peaks. This was the work of Billey, Burdzy and Sagan in their work titled *Permutations with given peak set*, here is the citation [1].

Having introduced you to the mathematics of the course, I now describe the structure of the summer school.

## Workshop Structure

Each day had a morning and an afternoon session both of which consisted of a lecture (60-90 minutes in length to be decided by the professor), a short break, and a problem solving session. These sessions were separated by a lunch at midday at the university's cafeteria.

As I wanted to cover a lot of material while also giving attendees small problems to solve while I lectured, I opted for the 90-minute lecture block, and we used every minute of that time! Students were highly engaged and often asked questions, which naturally led to generalizations of results I presented, and at times even open problems in the area. The problem solving sessions were led by two advanced graduate students. I provided more advanced and difficult problems for them to engage with the material and students spent about an hour working on those in small groups. At the conclusion of the working hour, one member would present their team's work on a selected problem to the rest of the workshop participants. Also the graduate students create lecture notes from the material covered both in the lecture, which are made available after the conference to more widely disseminate the lecture material presented.

One of my main objectives in attending Días de Combinatoria was to build further collaborations with students and faculty across Latin America. With this goal in mind, I spent the last lecture day introducing a variety of generalizations to the work I presented and allowed time for us to brainstorm new potential research problems. At the conclusion of the lecture time, we had developed a variety of open problems, and some initial investigation on these problems took place during the last problem solving session. I am hopeful that students are interested enough with the problems they developed that they will continue in their investigation and that I might have further opportunities to continue collaborating with colleagues in Colombia and in other Latin American countries.

## Parting Thoughts

I cannot stress enough what a great experience I had attending Días de Combinatoria and based on the high level of engagement and their positive comments at the end of the week, I can confidently say that students also had a great experience. I encourage more students and faculty members in the U.S. to look for opportunities like this one to engage with other Latin American countries and find collaborations beyond our border. I understand that travel funding might be limited, especially given the current status of scientific funding in our country, in which case you can connect to events virtually. To begin connecting, I recommend you attend the virtual programming organized by the International Community of Mathematicians from Latin America [3].

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## References

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