Sample MS Statistics Exam

Note: There are six problems. Partial credit will be given for partial solutions but scoring will emphasize completely correct answers. You are guaranteed a passing score if you solve at least four problems. Show all work and justify all claims.

1. Let $X$ and $Y$ be independent $N(0, 1)$ random variables. Show that $U = (X + Y)/\sqrt{2}$ and $V = (X - Y)/\sqrt{2}$ are also independent $N(0, 1)$ random variables.
   
   (Note: the $N(0, 1)$ density is $f(z) = ce^{-z^2/2}$.)

2. The $P(\lambda)$ distribution has pmf $p(x) = e^{-\lambda}x^x/x!$, for $x \in \{0, 1, \ldots\}$.
   
   (a) Find the characteristic function (or the moment generating function) of $X \sim P(\lambda)$.
   
   (b) If $X_1, \ldots, X_n$ are independent with $X_i \sim P(\lambda)$, prove that $\sum_{i=1}^n X_i \sim P(\sum_{i=1}^n \lambda_i)$. 

3. Let $\{X_i\}_{i \geq 1}$ be i.i.d. with $E(X_i) = 0$ and $V(X_i) = 2$. Find the limit of the sequence $\{Z_n\}_{n \geq 1}$ given by $Z_n = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}}$, specifying if the convergence is almost sure, in probability or in distribution.

4. Prove that if a decision rule is admissible and has constant risk, then it’s minimax.

5. Let $X_1, \ldots, X_n$ be i.i.d. with pdf $f(x; \theta) = \frac{\theta^x x e^{-\theta x}}{\Gamma(k)}, \ x > 0,$ for $k > 0$ fixed and known. Find the uniformly most powerful test of size $\alpha$ for the hypotheses $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$ (and prove it’s uniformly most powerful). Find explicitly all constants involved, if possible; if not possible, explain how to compute them numerically.

6. Let $X_1, \ldots, X_n$ be i.i.d. with $\beta(a, b)$ distribution, whose density is 

   \[ f(x; a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1 - x)^{b-1} I_{(0,1)}(x), \]

   with $a > 0$ and $b > 0$. If $a = kb$ with $k$ a known constant, find a sufficient and complete statistic for the parameter $\theta = b$ (and prove it’s sufficient and complete).