There are eight problems. Show all work. Be sure to justify all claims. Partial credit will be given for partial solutions, but scoring will emphasize completely correct answers. You are guaranteed a passing score if you successfully solve at least five problems.

(1) Let $G$ be a group and let $g \in G$.
   (a) Define the order of $g$.
   (b) Show that if $\phi$ is an automorphism of $G$, then $g$ and $\phi(g)$ have the same order.
   (c) Show that if $a, b \in G$, then $ab$ and $ba$ have the same order.

[Alternative: What is the largest possible order of an element of the alternating group $A_8$? Give an example of an element with that order.]

(2) Let $T : V \rightarrow V$ be a linear transformation and let $x, y, z$ be eigenvectors for $T$ with respective eigenvalues $\lambda, \mu, \nu$. Prove that if $\lambda, \mu, \nu$ are distinct, then $x, y, z$ are linearly independent.

[Alternative: Let $A$ be a real symmetric matrix. Prove that $A^2 + I$ is invertible.]

(3) Factor the following polynomials into irreducible factors in the given rings.
   (a) $x^6 - 2016$ in $\mathbb{Q}[x]$.
   (b) $x^4 - 1$ in $\mathbb{Z}_{13}[x]$.
   (c) $x^9 + 1$ in $\mathbb{Z}_3[x]$.

*Hint for (c):* cubes have special properties in rings of characteristic 3. If you don’t know this fact, you probably should not spend time working on this part of the problem.

[Alternative: Let $(x^2 - 3)$ be the principal ideal of $\mathbb{Z}[x]$ generated by $x^2 - 3$ and let $\mathbb{Z}[\sqrt{3}] = \{ a + b\sqrt{3} \mid a, b \in \mathbb{Z} \}$. Prove that $\mathbb{Z}[x]/(x^2 - 3) \cong \mathbb{Z}[\sqrt{3}]$.]

(4) Prove that every group of order 30 has a normal subgroup of order 3 or 5.

[Alternative: Let $G$ be a finite group and $p$ a prime divisor of $|G|$.
   (a) Prove that a normal $p$-subgroup of $G$ is contained in every $p$-Sylow subgroup of $G$.
   (b) Prove that the intersection of all of the $p$-Sylow subgroups of $G$ is the largest normal $p$-subgroup of $G$. (We consider a trivial group a $p$-group in this part.)]

(5) Let $V_n$ be the vector space of polynomials of degree at most $n$ over $\mathbb{C}$. For $f, g \in V_n$, define $\langle f, g \rangle = \frac{1}{2} \int_{-1}^{1} f(x)g(x) \, dx$.
   (a) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on $V_n$.
   (b) Find an orthonormal basis for $V_3$.

[Alternative: Define a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ by $\langle (x_1, x_2), (y_1, y_2) \rangle = 5x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$. It’s easy to see that $\langle \cdot, \cdot \rangle$ is bilinear and symmetric. Don’t prove this.
   (a) Show that $\langle \cdot, \cdot \rangle$ is positive definite, i.e., that $\langle (a, b), (a, b) \rangle > 0$ if $(a, b) \neq (0, 0)$.
   (b) Find an orthonormal basis for $\mathbb{R}^2$ with respect to this form.]
(6) In this problem, we assume \( \phi(1) = 1 \) for a ring homomorphism \( \phi \). If \( d \) is an integer, we define \( \mathbb{Z}[\sqrt{d}] = \{ a + b\sqrt{d} \mid a, b \in \mathbb{Z} \} \). This is an integral domain.

Suppose \( p, q \) are distinct prime integers. Prove that there is no ring homomorphism \( \phi : \mathbb{Z}[\sqrt{p}] \to \mathbb{Z}[\sqrt{q}] \).

[Alternative: Show that there are no integers \( x, y, z \) with \( x^2 + y^2 + z^2 = 9 \cdots 9 \) as long as there are at least three 9s.]

(7) State and prove the First Isomorphism Theorem for groups.
[Alternative: State and prove the First Isomorphism Theorem for rings.]

(8) Let \( V, W \) be vector spaces over a field \( F \) and let \( \phi : V \to W \) be a linear map. Let \( v_1, \ldots, v_n \) be distinct elements of \( V \).

Show that if \( \phi(v_1), \ldots, \phi(v_n) \) form a basis of \( W \), then \( v_1, \ldots, v_n \) are linearly independent and \( \phi \) is surjective.

[Alternative: Let \( V \) be a finite dimensional vector space and let \( A, B \) are subspaces of \( V \). Prove that \( \dim A \cap B + \dim(A + B) = \dim A + \dim B \).]